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Note

A Counterexample in the Theory of Best Approximation

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We give an example of a domain Ω with smooth boundary and with compact subsets K_1 and K_2, such that K_1 and K_2 have disjoint hulls, but such that there is no function u, harmonic on Ω, which is negative on K_1 and positive on K_2.

Let $\Omega \subset \mathbf{R}^d$ be a bounded connected open set and let $f: \bar{\Omega} \to \mathbf{R}$ be continuous. A standard problem in approximation theory is to find a function u, harmonic on Ω, such that

$$\sup_{z \in \Omega} |f(z) - u(z)| \equiv \|f - u\|_\infty$$

is as small as possible. Such a u is called a *best harmonic approximation to f.*

A normal families argument shows that best harmonic approximations always exist. What approximation theorists look for are simple tests which will determine whether a given u is a best approximation to a given f.

A standard best-approximation test is stated in terms of the "hulls" of certain compact subsets of $\bar{\Omega}$ [1]. If $K \subset \bar{\Omega}$ is compact, we define the hull of K (denoted $\hat{K}$) to be the union of K along with all of the components of $\mathbf{R}^d \setminus K$ which are completely contained in Ω. Roughly speaking, $\hat{K}$ is what you get by filling in K's holes. But you have to be a little careful when Ω is not simply connected. If $\Omega = \{z \in \mathbf{C} : 0 < |z| < 1\}$ and $K = \{z : |z| = \frac{1}{2}\}$, then $\hat{K} = K$, because K's "hole" touches $\partial\Omega$.

Now, let f be as above, and suppose that u is harmonic on Ω and continuous on $\bar{\Omega}$. Let $\rho = \|f - u\|_\infty$ and set

$$K_+ = \{z \in \bar{\Omega} : f(z) - u(z) = +\rho\}$$
$$K_- = \{z \in \bar{\Omega} : f(z) - u(z) = -\rho\}.$$

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An easy argument involving the maximum principle shows that if $K_+ \cap \hat{K}_-$ or $K_- \cap \hat{K}_+$ is non-empty, then u is a best harmonic approximation to f. It turns out that for reasonable domains without holes, this "linking" condition on K_+ and K_- is also necessary for u to be a best approximation. The proof of this fact makes use of various "Runge-type" theorems. If $\hat{K}_+ \cap \hat{K}_- = \varnothing$, then, under suitable hypotheses on Ω, one can build a function ϕ which is harmonic on Ω, continuous on $\bar{\Omega}$, and which is positive on K_+ and negative on K_-. One then adds a small scalar multiple of ϕ to u to get a better approximation.

It was asked whether such a simple linking condition might characterize best approximations on domains Ω which are not simply connected but whose boundaries are not especially pathological. The question, in its mildest form, boils down to this: *Let $\Omega \subset \mathbf{R}^d$ have a smooth boundary. Let K_1 and K_2 be compact subsets of Ω such that $\hat{K}_1 \cap \hat{K}_2 = \varnothing$. Does there always exist a ϕ harmonic on Ω such that $\phi > 0$ on K_1 and $\phi < 0$ on K_2?*

The reason that the answer to the question is not obviously "yes" is that the hull of $K_1 \cup K_2$ will generally be larger than that of $\hat{K}_1 \cup \hat{K}_2$, making the use of a Runge-type theorem impossible.

It turns out that the answer is **NO**.

We give the counterexample in $d = 2$; it extends, with trivial modifications, to higher dimensions.

THEOREM 1. *Let $\Omega = \{z \in \mathbf{C} : 1 < |z| < 2\}$. Let $K_1 = \{z : |z| = 1.1\} \cup \{1.6\}$ and $K_2 = \{z : |z| = 1.9\} \cup \{1.5\}$. There is no ϕ which is harmonic on Ω, negative on K_1, and positive on K_2.*

Remark. Note thet $\hat{K}_i = K_i$ for $i = 1, 2$, while the hull of $K_1 \cup K_2$ is $\{z : 1.1 \leqslant |z| \leqslant 1.9\}$.

Proof. Suppose that such a ϕ exists. We can symmetrise ϕ to make it even in the y-variable. The point 1.5 must be connected to $\partial\Omega$ by a path γ which lies completely inside the set $\{z : \phi(z) > 0\}$. (The fact that γ might have wild behavior near the boundary is irrelevant.) Since γ cannot meet $\{z : |z| = 1.1\} \subset K_1$, it must pass through $\{z : |z| = 1.9\}$. Because of ϕ's symmetry, the complex conjugate of γ must have the same property. The union of these two paths, plus the circle $\{z : |z| = 1.9\}$, is a subset of $\{z : \phi(z) > 0\}$, which completely surrounds—in Ω—the point 1.6, and this is impossible if $\phi(1.6) < 0$. Q.E.D.

COROLLARY 2. *The linking condition $\hat{K}_+ \cap \hat{K}_- \neq \varnothing$ is not necessary for best approximation in an annulus.*

Proof. Let Ω, K_1, and K_2 be as above. Let f be any function continuous on $\bar{\Omega}$ such that $f \equiv -1$ on K_1, $f \equiv 1$ on K_2, and $|f| < 1$ elsewhere. Then the zero function is a best approximation to f, because any better harmonic approximation would have to be negative on K_1 and positive on K_2, which is impossible; and this holds even though $K_- = K_1$ and $K_+ = K_2$ are not "linked." Q.E.D.

The proof of Theorem 1 has an amusing corollary:

COROLLARY 3. *Let u be harmonic on $\Omega = \{z \in \mathbf{C} : |z| < b\}$, continuous on $\bar{\Omega}$, and satisfy*

$$u(z) < 0 \qquad |z| = a$$

$$u(z) > 0 \qquad |z| = b.$$

Then the set $\{z \in \Omega : u(z) < 0\} \cup \{z : |z| \leqslant a\}$ is star-like with respect to the origin.

REFERENCE

1. W. K. HAYMAN, D. KERSHAW, AND T. J. LYONS, The best harmonic approximant to a continuous function, *in* "Anniversary Volume on Approximation Theory and Functional Analysis," pp. 317–237, ISNM 65, Academic Press, New York, 1984.